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**MULTIDIMENSIONAL HYPERGEOMETRIC FUNCTIONS  
AND THEIR APPLICATION IN SOLVING BOUNDARY  
VALUE PROBLEMS FOR DEGENERATE SECOND  
ORDER PARTIAL DIFFERENTIAL EQUATIONS**

**ABSTRACT**

of the dissertation work for  
the Philosophy Doctor degree (PhD)  
in specialty 6D060100 – «Mathematics»

**Relevance of the research topic.** Hypergeometric functions occupy a special place in the class of special functions. Often solutions to various problems of mathematics, physics, mechanics, economics are expressed by means of definite integrals containing special functions. But it is not always possible to solve these integrals using well-known algorithms, since the integrands often depend on one or more parameters. This leads to the calculation of a huge number of complex integrals. In such cases, the capabilities of modern computer technology and numerical methods are not enough. Then it becomes appropriate to represent the integral in terms of a convergent infinite series or product.

The study of the properties of hypergeometric functions is of great importance, since hypergeometric functions are widely used in the study of the solvability of boundary value problems for partial differential equations. In this dissertation research, a four-dimensional degenerate second-order partial differential equation of elliptic type is considered, fundamental solutions are constructed for it, solutions of boundary value problems are constructed, which are expressed by four-dimensional hypergeometric functions. Using transformations and the operator method, the hypergeometric series of four arguments are represented as the product of several hypergeometric series depending on fewer variables and parameters. To solve boundary value problems for degenerate elliptic equations in a finite domain, it is required to construct the Green's function, which is represented as a fundamental solution. Thus, the study of the properties of hypergeometric functions of four variables and their application in the study of the solvability of boundary value problems for degenerate elliptic equations determine the relevance of this topic of the thesis.

The survey and analysis of the study degree of the issues considered in the proposed dissertation showed that the development of the theory of hypergeometric functions has progressed very much over the past century. Hypergeometric functions are used in solving differential equations of a huge spectrum of problems, for example, they are used in mathematical physics, heat conduction problems and some sections of celestial mechanics, building potentials, elements of mathematical statistics, the study of electromagnetic oscillations, and aerodynamics, theories of communication and superstrings; they are also found in

applied problems of quantum chemistry and gas dynamics, applications of quantum field theory and astronomy. The properties of hypergeometric functions of several variables are studied; for a number of functions, formulas for the integral representation, expansion formulas, adjacent relations and analytic continuation are obtained.

**The purpose of the dissertation work** is to construct fundamental solutions of the Gellerstedt equation of four variables and to study the correct solvability of boundary value problems.

**The objects of the study** are hypergeometric functions of four variables and a degenerate Gellerstedt equation of elliptic type.

**The subject of the study** is the use of hypergeometric functions of four variables in the construction of solutions to boundary value problems for a degenerate equation ( $H$ ) of elliptic type.

**The objectives of the dissertation research:**

- obtaining linearly independent solutions of differential systems of equations for a number of hypergeometric functions of four variables;
- obtaining expansion formulas for hypergeometric functions of four variables;
- construction of fundamental solutions of a degenerate elliptic equation in  $R_+^4$ ;
- statement and study of solvability questions of boundary value problems for an equation ( $H$ ) with degenerations in an unbounded domain;
- statement and study of problems for an equation ( $H$ ) with mixed Dirichlet and Neumann conditions in an unbounded domain;
- solution of boundary value problem  $N$  for a degenerate equation ( $H$ ) in a bounded domain.

**Provisions to be defended:**

1) Linear independent solutions of differential systems of equations for some hypergeometric functions of four variables are constructed.

2) Operator identities and expansion formulas for some hypergeometric Gauss series of four arguments are proved by means of inverse pairs of Burchnall-Chaundy  $\nabla_{x,y}(c)$  and  $\Delta_{x,y}(c)$ , Hasanov-Srivastava  $\tilde{\nabla}_{x,y,z,t}(c)$  and  $\tilde{\Delta}_{x,y,z,t}(c)$ , Choi-Hasanov  $H(a,b)$  and  $\bar{H}(a,b)$  operators.

3) Sixteen fundamental solutions of the four-dimensional degenerate elliptic equation are constructed in explicit form.

4) A method for constructing boundary value problems for the four-dimensional generalized Gellerstedt equation in an infinite domain is developed. The uniqueness and existence theorems of the solution of boundary value problems are proved.

5) The uniqueness theorem for solving the problem  $N$  in a finite domain is proved. The Green's function of the problem  $N$  is constructed. An explicit solution to the problem  $N$  is obtained.

**Research methods.** In the course of implementing the goal and objectives of the dissertation research, the qualitative properties of hypergeometric series of many variables, classical methods of partial differential equations, methods of integral calculus, operational methods, the method of energy integrals and the extremum principle, the Green's function method, the Mellin-Barnes transformation method, operator methods of Burchnall-Chaundy and Choi-Hasanov.

**The scientific novelty of the study.**

1. For a number of four-dimensional hypergeometric functions systems of differential hypergeometric equations are solved, linearly independent solutions of the corresponding systems are found in explicit form.

2. For some hypergeometric functions of four variables are obtained decomposition formulas by using various operators.

3. For the first time fundamental solutions of the degenerate elliptic Gellerstedt equation of four arguments are constructed, through which a number of boundary value problems in an unbounded domain are solved. In a bounded domain, problem  $N$  is formulated, the uniqueness of the solution for problem  $N$  is proved, the Green's function of the problem is constructed, and the existence of an explicit solution to this problem is proved.

**The validity and reliability** is confirmed by the publication of the main results in the journals included in the international scientometric bases; this is also confirmed by the use of well-known classical methods used in the theory of partial differential equations; consistent justification and proof of each obtained result; to work with hypergeometric functions of several variables, proven algorithms and methods are used, verified and published in fundamental works of the theory of hypergeometric functions.

**Theoretical and practical significance of the research.** The topic of the dissertation research has a theoretical orientation. Results related to the study of the properties of hypergeometric functions will add to the knowledge base of the special functions theory. The rest of the results obtained can be applied in the theory of boundary value problems for elliptic equations with degenerations.

**Relation of dissertation work with other research works.** The proposed dissertation is carried out within:

1) the draft program of grant financing of fundamental and applied scientific research of the MES of the Republic of Kazakhstan for 2018-2020 "Mathematical modeling of the dynamics of elastic-deformable porous media taking into account the frequency dependence of the coefficient of friction (with memory)" No. AP05131026;

2) grant funding of the Abai Kazakh National Pedagogical University for 2020 "Development of the methods for constructing solutions to boundary value problems for four-dimensional degenerate equations of elliptic type" Contract No. 3 of 05.01.2020.

**Approbation of the work.** During the research on the dissertation topic, the obtained intermediate results were discussed and reported at seminars and conferences. Abstracts and articles were published in collections of materials of 14

scientific conferences, of which 11 had international status: International scientific conference «Differential equations and related problems» (2018, Sterlitamak, RB); Fourteenth international Asian school-seminar «Problems of optimization of complex systems» (2018, Issyk-Kul lake, KR); The VI International scientific conference «Modern problems of the applied mathematics and information technology – Al-Khorezmiy 2018» (2018, Tashkent, RU); III International scientific and practical conference «Computer science and applied mathematics» (2018, Almaty, RK); VIII International scientific and methodological conference «Mathematical modeling and information technologies in education and science» (2018, Almaty, RK); V International scientific conference «Non-local boundary value problems and related problems of mathematical biology, computer science and physics» (2018, Nalchik, RKB); International conference Voronezh spring mathematical school The Pontryagin reading-XXX «Modern methods of the theory of boundary value problems» (2019, Voronezh, RF); Scientific seminar «Problems of applied mathematics and computer science» (2019, Aktobe, RK); International scientific conference «Inverse problems in finance, economics and life sciences» (2019, Almaty, RK); IV International scientific and practical conference «Computer science and applied mathematics» (2019, Almaty, RK); Uzbek-Russian scientific conference «Non-classical equations of mathematical physics and their applications» (2019, Tashkent, RU); International scientific conference «Modern problems of differential equations and related branches of mathematics» (2020, Fergana, RU); International conference Voronezh spring mathematical school The Pontryagin reading-XXXI «Modern methods of the theory of boundary value problems» (2020, Voronezh, RF); Republican Scientific Conference with the participation of foreign scientists «Non-classical equations of mathematical physics and their applications» (2020, Tashkent, RU).

**Publications.** The results on the dissertation topic were issued in 21 publications, of which 3 articles were published on high peer-reviewed international journals, 3 - in journals recommended by the CCSES of the RK MES, 1 - in a journal included in the RSCI database, 14 theses and articles published in the collections of the results of international conferences.

Publications high peer-reviewed scientific journals:

1. Hasanov A., Berdyshev A.S., Ryskan A. Fundamental solutions for a class of four-dimensional degenerate elliptic equation // *Complex Var. Elliptic Equ.* –2020. –Vol. 65, No. 4, –P. 632–647.

2. Berdyshev A.S., Ryskan A. The Neumann and Dirichlet problems for one four-dimensional degenerate elliptic equation // *Lobachevskii J. Math.* –2020. – Vol. 41, No. 6, –P. 1051–1066.

3. Berdyshev A.S., Hasanov A., Ryskan A.R. Solution of the Neumann problem for one four-dimensional elliptic equation. // *Eurasian Math. J.* –2020. – Vol.11, No. 2. –P. 93–97.

Publications in journals recommended by the CCSES of the RK MES:

1. Рыскан А.Р. Формулы разложения с операторами  $H$  Гипергеометрических рядов Гаусса от четырех переменных второго порядка // *Вестник КазНПУ им.Абая. Сер. Физ.-мат. науки.* –2020. №3(71). –С. 79–84.

2. Berdyshev A.S., Hasanov A., Ryskan A.R. Decomposition formulas for some quadruple hypergeometric series // *Bullet. Karaganda Univ. Math. ser.* – 2020. No. 4 (100), –P. 43–54.

3. Рыскан А.Р. Решение задачи Дирихле для вырождающегося эллиптического уравнения второго порядка // *Вестник КазНПУ им.Абая. Сер. Физ.-мат. науки.* –2019. № 4 (68). –С. 92–98.

**The structure and scope of the thesis.** The dissertation work is presented on 107 pages, consists of an introduction, four chapters with sections, a conclusion and a list of used sources.

**The main content of the thesis.** The introduction provides a brief chronology of the stages in the development of the theories of degenerate elliptic equations and hypergeometric functions, describes the current state of the object and subject of this study. The relevance and novelty of the topic of the dissertation at the current moment are stated, the goal is set and the tasks of the work are formulated, the methods of research, the validity and reliability of the scientific provisions, conclusions and recommendations formulated in the dissertation, theoretical and practical significance, testing and publication of the main results are indicated.

The first section is divided into four subsections. The first subsection provides well-known definitions, properties, formulas and preliminary information about hypergeometric functions. In the second subsection, the first results of the dissertation research are presented: the definitions of systems of second order partial differential equations for a number of four-dimensional hypergeometric functions are given and their linearly independent solutions are found. In the third and fourth subsections, operator identities and formulas for the expansion of hypergeometric functions of four variables with the use of different operators are presented, the validity of the operator identities is proved using the Mellin-Barnes transformation, and the validity of the decomposition formulas is proved by means of the operator identities.

The second chapter is devoted to the construction of fundamental solutions of a second-order differential equation with four hyperplanes of degeneracy. Sixteen fundamental solutions are obtained, which are expressed by Lauricella hypergeometric functions. Also, a theorem on the singularity of fundamental solutions is proved, where the formulas for differentiation, autotransformation and an expansion formula are used that represent the hypergeometric function of several variables as the product of Gaussian functions of one variable.

In the third chapter, boundary value problems for an elliptic equation with several degenerations are formulated in an infinite domain, proofs of existence and uniqueness theorems for solutions of the posed problems are given, and their solutions are written out in explicit form. The uniqueness theorem for the solution of problems with Dirichlet boundary conditions is proved by using the extremum principle. To prove the uniqueness of the solution to the boundary value problem  $N^\infty$ , the energy integral method are used.

The fourth chapter is devoted to solving a boundary value problem  $N$  for a degenerate equation  $(H)$  in a bounded domain. A proof of the uniqueness theorem for the boundary value problem  $N$  is given. The Green's function of problem  $N$  is constructed. An existence theorem for a solution to Problem  $N$  is proved, the solution is written explicitly.

In the conclusion, the main results obtained during the dissertation research are presented and conclusions are drawn on them.